Journal of Statistical Physics, Vol. 76, Nos. 5/6, 1994

On Quantum Stability for Systems under Quasiperiodic Perturbations

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Received October 26, 1993; final March 1, 1994

We prove that the spectrum defined in terms of the autocorrelation function of a harmonic subject to a quasiperiodic perturbation, is, at resonance, transient absolutely continuous, covering the whole line. In the nonresonant case, and under some supplementary Diophantine condition, it is pure point, coinciding with the spectrum of a special almost-periodic function.

KEY WORDS: Quantum stability; quasiperiodic systems; spectrum; autocorrelation function; quantum chaology.

1. INTRODUCTION AND SUMMARY

In an interesting paper, Bunimovich *et al.*[1] considered a harmonic oscillator subject to several types of time-dependent periodic and quasiperiodic perturbations (see also ref. 2 for the periodic case). Such models are the simplest of a class of quantum systems in external time-varying fields (periodic or quasiperiodic), which have been extensively studied in connection with "quantum chaology" (see ref. 3 for periodic and refs. 4–10 for quasiperiodic perturbations). In the quasiperiodic examples they looked at, Bunimovich *et al.*[1] used the growth of the expectation value of the kinetic energy as a spectral indicator. This quantity does indeed distinguish between different types of classical behavior from the point of view of ergodic theory (e.g., diffusive growth $\simeq t$ in the case of K-systems, and boundedness in the case of classically ergodic but not mixing systems), but its connection with the natural notion of spectrum defined in terms

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0022-4715/94/0900-1479$07.00/0

of the autocorrelation function[11] is not clear [in the *periodic* case, if the spectrum of the unperturbed operator is discrete, and if the Floquet spectrum (ref. 12, Chapter 7) of the perturbed operator is *continuous*, the expectation value of the kinetic energy grows with time t (ref. 13, Theorem 7), but the converse does not necessarily hold]. It seemed to us therefore interesting to study the spectrum in the sense of ref. 11—which we call the *autocorrelation spectrum*—of models described by Hamiltonians

$$H(t) = H_0 + V(t) \tag{1.1}$$

where

$$H_0 = \omega_0 a^+ a \tag{1.2}$$

and

$$V(t) = (a^+ + a)[\lambda_1 \cos(\omega_1 t) + \lambda_2 \cos(\omega_2 t)] \tag{1.3}$$

where a, a^+ are the usual annihilation and creation operators of one boson (harmonic oscillator) on Fock space $\mathscr{F}$. Let $\psi(\cdot)$ be the solution of the Schrödinger equation

$$i\frac{\partial\psi(t)}{\partial t} = H(t)\,\psi(t) \tag{1.4}$$

and with initial value

$$\psi(0) = \psi \tag{1.5}$$

Define[11] the autocorrelation function $C_\psi(t)$ by

$$C_\psi(t) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \langle \psi(s), \psi(s+t)\rangle\, ds \tag{1.6}$$

where $\langle \cdot, \cdot \rangle$ denotes scalar product on $\mathscr{F}$, conjugate linear in the first (left) vector. Assume for the moment that the limit on the r.h.s. of (1.6) exists. Then C_ψ is positive-definite, and may therefore be expressed as

$$C_\psi(t) = \int e^{itE}\, d\mu_\psi(E) \tag{1.7}$$

for some positive Stieltjes measure μ_ψ, by Bochner's theorem.[14] The *autocorrelation spectrum* of $H(t)$ is[11] the support of the measure μ_ψ. In the time-independent (resp. periodic) case, this notion agrees with the usual definition (resp. definition of Floquet spectrum). As usual, the $\psi \in \mathscr{F}$ such that μ_ψ is absolutely continuous (a.c.), singular continuous (s.c.), and pure point (p.p.) define subspaces $\mathscr{F}_{\text{a.c.}}$, $\mathscr{F}_{\text{s.c.}}$, and $\mathscr{F}_{\text{p.p.}}$, respectively.

In Section 2 we consider separately the two cases (a) resonant case $\omega_0 = \omega_1$ (with ω_2 incommensurate with ω_0), and (b) nonresonant case (ω_0, ω_1, ω_2 incommensurate).

We prove that at resonance the autocorrelation spectrum is transient absolutely continuous[15] and covers the whole line, while in the incommensurate case the spectrum is pure point under some supplementary Diophantine condition and coincides with the spectrum (ref. 16, Chapter VI, pp. 155ff.) of a special almost-periodic function. The basic elements in the proof are the theorems (Appendix A) relating the decay of the autocorrelation function with the type of spectral measure, and the explicit form of the evolution operator acting on a coherent state, which allows an explicit evaluation of the limit on the r.h.s. of (1.6), as well as extension of the result to other coherent states by the group property of the latter. It should be remarked that Combescure[17] succeeded in finding the autocorrelation spectrum of a two-level system with a perturbation related to the Thue–Morse sequence, a much more difficult problem. The perturbation in this case is, however, aperiodic, in contrast to the quasiperiodic one we consider.

A much more difficult problem, related to the model of Combescure,[17] is described by the Hamiltonian

$$\tilde{H} = \tilde{H}_0 + \tilde{V}(t) \tag{1.8}$$

with

$$\tilde{H}_0 = V\sigma_z \tag{1.9}$$

and

$$\tilde{V}(t) = \lambda f(t)\,\sigma_x \tag{1.10}$$

where σ_x, σ_z are Pauli matrices on $\mathbb{C}^2$ and f is quasiperiodic.

This model is physically more interesting as a model of a two-level atom subject to an external quasiperiodic (e.g., bichromatic) electric field.[5,7–10]

In ref. 5 many results concerning this model (and its N-level generalization) were presented. Using a generalized quasienergy operator[13] or the generalized Floquet operator[4] (whose spectral properties are equivalent to those of the quasienergy operator as well as to the autocorrelation spectrum used in this paper) the authors succeeded in proving the existence of pure point as well as continuous spectrum in several special cases. In particular they showed the stability of the pure point spectrum for small perturbations. The point spectrum is associated with stable quasiperiodic behavior[4,13] and the continuous spectrum is a sign of instability, akin to (unstable) chaotic behavior (in the atomic case, continuous spectrum corresponds to "weaker localization" and hence less "quantum suppression of classical chaotic diffusion"[3]).

The results for the oscillator model (1.1)–(1.3) concerning the point spectrum were also obtained in ref. 6, where the continuous spectrum was also treated from a "generic" point of view. The present paper gives more precise information for the resonant case, where the spectrum is proved to be transient absolutely continuous. Moreover, the techniques used here to deduce spectral properties from the decay of correlation functions are of completely different nature, and we feel that they may be applicable to other problems. Finally, we refer to the beautiful review by Howland.[18]

2. THE OSCILLATOR PROBLEM

Solution of (1.1)–(1.5) may be found explicitly in the interaction picture[19] in the special case where ψ is a (Glauber) coherent state. For clarity, and because the result is crucial, we briefly recall the derivation. Let

$$f(t) \equiv \lambda_1 \cos(\omega_1 t) + \lambda_2 \cos(\omega_2 t) \tag{2.1}$$

In the interaction picture $\psi(t) = e^{-itH_0}\psi_I(t)$, the equation for ψ_I is

$$i\frac{\partial \psi_I}{\partial t} = V_I(t)\,\psi_I(t), \qquad \psi_I(0) = \psi \tag{2.2}$$

where

$$V_I(t) = e^{itH_0}V(t)\,e^{-itH_0} = f(t)(ae^{-i\omega_0 t} + a^+ e^{i\omega_0 t})$$

Writing

$$\beta(t) \equiv -if(t)\,e^{i\omega_0 t} \tag{2.3}$$

we may rewrite the equation for ψ_I in the form

$$\frac{\partial \psi_I}{\partial t} = [\beta(t)\,a^+ - \bar{\beta}(t)\,a]\,\psi_I(t) \tag{2.4}$$

Since the Hamiltonian is linear in the operators of the Weyl–Heisenberg group, the time-evolution operator U_t, defined by $\psi_t(t) = U_t(t)\,\psi_t(0)$, is an operator of the group representation, i.e.,

$$U_t(t) = e^{-i\varphi(t)} D(\gamma(t)) \tag{2.5}$$

where φ is a real phase, γ a complex-valued function of t, and

$$D(\alpha) = e^{\alpha a^+ - \bar{\alpha} a} \tag{2.6}$$

Hence, if the initial state is a (Glauber) coherent state

$$|\alpha\rangle = D(\alpha)\,|0\rangle \tag{2.7}$$

where $|0\rangle$ is the vacuum (ground state of the harmonic oscillator), it will remain coherent for all time, and therefore a solution exists of the form

$$\psi_t(t) = e^{-i\varphi(t)}\,|\alpha(t)\rangle \tag{2.8}$$

In particular, the expectation value of a in ψ_t is

$$\langle \psi_t |\, a\psi_t \rangle = \alpha(t) \tag{2.9}$$

Differentiating (2.9) and using (2.4), we obtain

$$\frac{d\alpha}{dt} = \beta(t), \qquad \alpha(t) = \alpha_0 + \int_0^t \beta(t')\,dt' \tag{2.10}$$

The phase may also be found;[19] in our case, it is zero. We now restrict ourselves to the special case $\alpha_0 = 0$ in (2.10), i.e., where the initial state is the vacuum $|0\rangle$. We further omit the subscript ψ in $C_\psi(t) = C_{|0\rangle}(t)$. In this case, (2.8), (2.10), and the alternative formula

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^+} e^{-\bar{\alpha} a}$$

for (2.6) yield for the autocorrelation function (1.6) the expression

$$C(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} ds \exp[I(s, t)] \tag{2.11}$$

where

$$I(s, t) = -\frac{|\alpha(s)|^2}{2} - \frac{|\alpha(s+t)|^2}{2} + e^{+it\omega_0}\,\overline{\alpha(s)}\,\alpha(t+s) \tag{2.12}$$

and α is given by (2.10) (with $\alpha_0 = 0$), and β was defined in (2.1), (2.3). We now distinguish two cases: (a) $\omega_0 = \omega_1$, with ω_2 and ω_1 incommensurate (resonant case); (b) ω_0, ω_1, and ω_2 incommensurate (nonresonant case).

2.1. Resonant Case

The $I(s, t)$ given by (2.12) is a rather long expression, but it may be expressed in the form (see Appendix C)

$$I(s, t) = -\frac{\lambda_1^2 s^2}{2}(1 - e^{it\omega_0}) - \frac{\lambda_1^2 t^2}{2} - \frac{\lambda_1^2}{2}(1 - e^{it\omega_0}) + sf_0(s, t) + tf_1(s, t) + f_2(s, t) \tag{2.13}$$

where for f_0, both f_1 and f_2 are continuous functions of both arguments, uniformly bounded in s, t

$$|f_0(s, t)| \leqslant M_0, \qquad |f_1(s, t)| \leqslant M_1, \qquad |f_2(s, t)| \leqslant M_2 \tag{2.14}$$

and such that

$$\operatorname{Re} f_0\left(\frac{2\pi n}{\omega_0}, t\right) = 0 \tag{2.15}$$

for any integer n. The appearance of quadratic terms in s and t in (2.13) is crucial and is only due to the resonance condition. By (2.13) and (2.14) we have

$$\begin{aligned} &\left|\int_{-T}^{T} ds \exp[I(s, t)]\right| \\ &\quad \leqslant \exp\left(-\frac{\lambda_1^2}{2} t^2 + tM_1 + M_2\right) \\ &\qquad \times \int_{-T}^{T} ds \exp\left[-\frac{\lambda_1^2 s^2}{2}(1 - \cos t\omega_0) - \frac{\lambda_1^2 st}{2}(1 - \cos t\omega_0) + sg(s, t)\right] \end{aligned} \tag{2.16}$$

where

$$g(s, t) \equiv \operatorname{Re} f_0(s, t) = \gamma(s, t) \sin \omega_0 t + \delta(s, t)(1 - \cos \omega_0 t) \tag{2.17}$$

(see Appendix C). In (2.17), $\gamma(s, t)$ and $\delta(s, t)$ are linear combinations of trigonometric functions [see again (C.3) of Appendix C] and thus

$$\sup_{s,t} |\gamma(s, t)| = M < \infty, \qquad \sup_{s,t} |\delta(s, t)| = N < \infty \tag{2.18}$$

We now have two cases:

(i) $t=2\pi n/\omega_0$, n integer. Using (2.15) and (2.17), we find, by (2.11) and (2.16),

$$|c(t)| \leqslant a \exp\left(-\frac{\lambda_1^2 t^2}{2}+bt\right) \tag{2.19}$$

for a, b positive constants, independent of t.

(ii) $t \neq 2\pi n/\omega_0$, for all integers n.

What may happen here is that, although $t \neq 2\pi n/\omega_0$, $t=2\pi n/\omega_0 + o(n^{-1})$. We write the second integral in (2.16) as $\int_0^T(\cdots)+\int_{-T}^0(\cdots)$. We estimate the first one; the second is similar.

The integral may be written in the form

$$I \equiv \int_0^T \exp[a(t)\, s^2 + b(t, s)\, s]\, ds \tag{2.20}$$

where

$$a(t) \equiv -\frac{\lambda_1^2}{2}(1-\cos\omega_0 t) \tag{2.21a}$$

$$b(t, s) \equiv g(t, s) - \frac{\lambda_1^2}{2} t(1-\cos\omega_0 t) \tag{2.21b}$$

Completing the square in (2.20), we obtain

$$I = \int_0^T ds \exp\frac{+b(t, s)^2}{4\,|a(t)|} \exp\left[-\frac{\lambda_1^2}{2}(1-\cos\omega_0 t)\left(s+\frac{b(t, s)}{2a(t)}\right)^2\right] \tag{2.22}$$

and hence

$$I \leqslant T \sup_{s\in[0, T]} \left\{\exp\left\{\frac{1}{2\lambda_1^2(1-\cos\omega_0 t)}\left[g(t, s)-\frac{\lambda_1^2}{2} t(1-\cos\omega_0 t)\right]^2\right\}\right\} \tag{2.23}$$

Now

$$\frac{1}{2\lambda_1^2(1-\cos\omega_0 t)}\left[g(t, s)-\frac{\lambda_1^2}{2} t(1-\cos\omega_0 t)\right]^2$$
$$=\frac{g(t, s)^2}{2\lambda_1^2(1-\cos\omega_0 t)}-\frac{t}{2}\, g(t, s)+\frac{\lambda_1^2}{8} t^2(1-\cos\omega_0 t) \tag{2.24}$$

and

$$\frac{g(t,s)^2}{2\lambda_1^2(1-\cos\omega_0 t)} = \frac{\gamma(s,t)^2 \sin^2\omega_0 t}{2\lambda_1^2(1-\cos\omega_0 t)} + \frac{\gamma(s,t)\,\delta(s,t)\sin\omega_0 t}{\lambda_1^2} + \frac{\delta(s,t)^2}{2\lambda_1^2}(1-\cos\omega_0 t) \tag{2.25}$$

The first term on the r.h.s. of (2.25) is the only "troublemaker," because of the small denominator. However, since $\sin^2\omega_0 t \simeq (t-2\pi n/\omega_0)^2$ and $1-\cos\omega_0 t \simeq \frac{1}{2}(t-2\pi n/\omega_0)^2$ for t close to $2\pi n/\omega_0$, no problem arises. Hence, by (2.18), (2.23), (2.24), and (2.25).

$$I \leqslant cT \exp\left[\frac{\lambda_1^2}{8} t^2(1-\cos\omega_0 t) + rt\right] \leqslant cT\exp\left(\frac{\lambda_1^2}{4} t^2 + rt\right) \tag{2.26}$$

for some positive constants c and r, independent of t, which implies by (2.11) and (2.16) the bound

$$|C(t)| \leqslant a \exp\left(-\frac{\lambda_1^2}{4} t^2 + bt\right) \tag{2.27}$$

for a, b positive constants independent of t.

By (2.19), (2.27) and a result of Sinha (Lemma A.1 of Appendix A), we thus have the following result.

Proposition 1. The autocorrelation spectrum in the resonant case is transient absolutely continuous, with support the whole real line $\mathbb{R}$.

2.2. Nonresonant Case

In this case the structure of the autocorrelation function is simpler and appears in Appendix D. The argument of the exponential is of the form $\sum_{j=1}^N u_j(t)\, v_j(s)$, where each $v_j(s)$ is a trigonometric function of type sine or cosine of one of the *arcs*: $2\omega_0 s$, $2\omega_1 s$, $2\omega_2 s$, $(\omega_0 \pm \omega_1)s$, $(\omega_0 \pm \omega_2)s$, $(\omega_1 \pm \omega_2)s$, with coefficients $u_j(t)$ which are almost periodic functions of t [linear combinations of sines and cosines of $(\alpha_j t)$, where α_j is linear in the frequencies $\omega_1, \omega_2, \omega_3$].

Above, N is an integer, independent of t. Expanding the exponentials of sines and cosines in terms of the modified Bessel functions, we find

$$C(t) = \lim_{t\to\infty} \frac{1}{2T}\int_{-T}^{T} ds \sum_{m_0,\ldots,m_N=-\infty}^{\infty} \prod_{i=0}^{N} I_{m_i}(u_i(t))\, e^{i\bar\omega\bar m s} \varepsilon_{\bar m} \tag{2.28}$$

where $\varepsilon_{\tilde{m}}$ assumes the values ± 1, and

$$\tilde{\omega}\tilde{m} \equiv \omega_0(2m_0 + \cdots + m_q) + \omega_1(2m_2 + \cdots + m_p) + \omega_2(2m_4 + \cdots + m_k) \tag{2.29}$$

where q, p, and k are integers between 1 and N. To simplify notation, we denote $m'_1 = 2m_0 + \cdots + m_q$, $m'_2 = 2m_2 + \cdots + m_p$, and $m'_3 = 2m_4 + \cdots + m_k$. We also assume that the frequencies $\omega_1, \omega_2, \omega_3$ satisfy "good Diophantine properties"

$$|\tilde{\omega}\tilde{m}| \geqslant C\,|\tilde{m}|^{-k} \tag{2.30}$$

for some $k \in \mathbb{N}$.

For the forthcoming proof any fixed k suffices, but only if $k \geqslant 4$ (taking k integer for simplicity) is the (three-dimensional Lebesgue) measure of the ω_1, ω_2, $\omega_3 \in \mathbb{R}^3$ which do not satisfy (2.30) zero (see, e.g., ref. 23). In (2.30), $|\tilde{m}| \equiv \sum_{i=1} |m'_i|$, and $0 < c < \infty$. Notice that

$$|I_{m_i}(u_i(t))| \leqslant \sqrt{\pi}\,\left|\frac{|u_i(t)|}{2}\right|^{m_i} \frac{e^{|u_i(t)|}}{|\Gamma(m_i + 1/2)|} \tag{2.31}$$

This estimate follows from (9.6.18), p. 376, of ref. 18, and hence the multiple sum in (2.28) converges.

Proposition 2. The autocorrelation spectrum in the nonresonant case, when the frequencies have "good Diophantine properties" (2.30), is pure point.

To prove the proposition we divide the sums in (2.28) into two parts:

$$C(t) = \lim_{T \to \infty} \frac{1}{2T} \sum_{-T}^{T} ds \left\{ \sum_{(m_0,\dots,m_N) \in A_1} + \sum_{(m_0,\dots,m_N) \in A_2} \right\} \times \prod_{i=0}^{N} I_{m_i}(u_i(t))\, e^{i\tilde{\omega}\tilde{m}s} \varepsilon_{\tilde{m}} \tag{2.32}$$

where

$$A_1 = \{(m_0,\dots, m_N) \,|\, \tilde{\omega} \cdot \tilde{m} = 0\} \tag{2.33a}$$

$$A_2 = \{(m_0,\dots, m_N) \,|\, \tilde{\omega} \cdot \tilde{m} \neq 0\} \tag{2.33b}$$

Correspondingly, we have

$$C(t) = f(t) + C_1(t) \tag{2.34}$$

where

$$f(t) \equiv \sum_{(m_0,\dots,m_N)\in A_1} \prod_{i=0}^{N} I_{m_i}(u_i(t))\, \varepsilon_{\tilde m} \tag{2.35}$$

and

$$C_1(t) = \lim_{t\to\infty} \frac{1}{T} \sum_{(m_0,\dots,m_N)\in A_2} \prod_{i=0}^{N} I_{m_i}(u_i(t)) \frac{\sin(\tilde\omega \tilde m T)}{\tilde\omega \tilde m}\, \varepsilon_{\tilde m} \tag{2.36}$$

Using now (2.30), we obtain

$$\left| \prod_{i=0}^{N} I_{m_i}(u_i(t)) \frac{\sin(\tilde\omega \tilde m T)}{\tilde\omega \cdot \tilde m}\, \varepsilon_{\tilde m} \right| \leqslant C \prod_{i=0}^{N} |I_{m_i}(u_i(t))| \left(\sum_{i=0}^{N} r_i\, |m_i| \right)^k \tag{2.37}$$

where r_i are integers independent of $(m_0,\dots, m_N)$ and T. By (2.37) and (2.31), the sum on the r.h.s. of (2.36) is finite and therefore

$$C_1(t) = 0 \tag{2.38}$$

Finally, we use Lemmas B1 and B2 of Appendix B. By Lemma B2 the almost periodic functions form a closed algebra in the L^∞-norm. Hence, it follows from (2.35), (2.31), and the fact that each u_i is almost periodic that f is almost periodic. Finally, (2.34), (2.35), (2.38), and Lemma B1 show that the concept of the spectrum of an almost periodic function, in this case of the special function f, agrees with the autocorrelation spectrum.

Hence, by Lemma B1, the autocorrelation spectrum is either pure point or empty. The latter occurs when $f \equiv 0$, which may be verified not to be the case. This concludes the proof of Proposition 2.

Theorem. In the resonant case $\mathscr{F} = \mathscr{F}_{\text{a.c.}}$, and in the nonresonant case $\mathscr{F} = \mathscr{F}_{\text{p.p.}}$.

Proof. We now come back to (2.10) and notice that the addition of the complex number α_0 does not change the structure of the autocorrelation function (2.11), (2.12). The same is true if ψ is a finite linear combination $\sum_{i=1}^{M} c_i\, |\alpha_i\rangle$, $M < \infty$. Since such are dense if i runs over a discrete set (e.g., the von Neumann lattice; see ref. 17 and references given there), and $\mathscr{F}_{\text{a.c.}}$ and $\mathscr{F}_{\text{p.p.}}$ are (closed) subspaces of $\mathscr{F}$, we have proved the theorem.

As a final remark to this section, we have shown that in both the resonant and the nonresonant cases the autocorrelation spectrum has qualitatively the same structure as the Floquet spectrum in the periodic case.[1,2] Further, in the resonant case, the support of C is "concentrated around" the sequence $t_n = 2\pi n/\omega_0$, which is also expected intuitively.

3. CONCLUSION

The present paper mentions only point spectrum and absolutely continuous spectrum, which may give the misleading impression that nothing else may occur in this context. However, from refs. 5, 6, and 13 one can expect that when ω_1/ω_2 is well approximated by rationals—e.g., a Liouville number—the spectrum is singular continuous. This is an open problem.

Several problems still persist for model (1.8)–(1.10), in particular, a rigorous analysis for large coupling. Counterexamples in ref. 5 show that results for small coupling cannot be extrapolated in general for large perturbations. It would thus be interesting to study the large-coupling limit of (1.8)–(1.10).

As a final remark, comparison with experiment is a delicate matter, because even the slightest damping may destroy all the "irregular" characteristics of chaotic dynamics in resonance fluorescence.[10]

APPENDIX A

In this appendix we present a lemma used in the text, which concern the decay of the Fourier transform of a Stieltjes measure, which may be found in a paper by Sinha.[21]

Lemma A1 (Ref. 21, Lemma 5, Appendix). Let f be the Fourier transform of a positive Stieltjes measure μ:

$$f(t) = \int e^{itE}\, d\mu(E) \tag{A.1}$$

and such that

$$f(t) = O(e^{-\beta |t|})$$

for some $\beta > 0$. Then μ is absolutely continuous, and has the whole real line $\mathbb{R}$ as its support.

By ref. 15, if $\mu = \mu_\psi$, the vector ψ is in the *transient* subspace of $\mathscr{F}$.

Remark. In a basic paper Sinha[22] proved that, if $f(t) = O(t^{-1/2-\varepsilon})$, $\varepsilon > 0$, then μ is absolutely continuous. This result is optimal, because there exist operators with purely s.c. spectrum, such that $f(t) = O(t^{-1/2+\varepsilon})$, with $\varepsilon > 0$ arbitrarily small.[22]

APPENDIX B

In this appendix we collect some results on almost-periodic functions used in the text. The basic reference is Kaltznelson.[16] The definition of almost periodic function, which we do not repeat here, is given in ref. 16, p. 155.

Let f be the Fourier transform of a Stieltjes measure, as in (A.1). Then, by a well-known theorem,[14]

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\, f(t)\, e^{-itE} = \mu(E+0) - \mu(E-0) \tag{B.1}$$

By (5.6), p. 161 of ref. 16, E belongs to the *spectrum* of the almost-periodic function f (defined in 5.9 of ref. 16, p. 159) if

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\, f(t)\, e^{-iEt} \neq 0 \tag{B.2}$$

By ref. 16, p. 162, the spectrum of an almost periodic function is countable (hence there is no continuous part). Therefore, by (B.1) and (B.2), the autocorrelation spectrum (defined in Section 1) coincides with the spectrum of C (the autocorrelation function) in case C is almost-periodic.

Lemma B1. Let the autocorrelation function C defined in Section 1 be almost-periodic and not identically zero. Then the autocorrelation spectrum is pure point and nonempty.

Proof. All assertions have been proved above, except for the last one, which says that the spectrum of the almost-periodic function f is nonempty if $f \not\equiv 0$. This follows from the uniqueness theorem (Theorem, p. 163 of ref. 16).

Let $AP(\mathbb{R})$ denote the set of all almost periodic functions on $\mathbb{R}$.

Lemma B2. $AP(\mathbb{R})$ is a closed subalgebra of $L^{\infty}(\mathbb{R})$.

Proof. See ref. 16, Theorem 5.7, p. 158.

APPENDIX C

The expression for $\alpha(s)$ in (2.10), (2.12) is

$$\alpha(s) = -i\frac{\lambda_1}{4\omega_0} e^{i2s\omega_0} - i\frac{\lambda_i s}{2} - \frac{\lambda_1}{4\omega_0} + \frac{\lambda_2\omega_0}{\omega_2^2 - \omega_0^2}$$
$$\times (\omega_0 e^{-is\omega_0} \cos s\omega_2 + \omega_2 e^{is\omega_0} \sin s\omega_2) \tag{C.1}$$

We are going to show that the function $f_0(s, t)$ given by (2.13) has the property $\mathrm{Re}(f_0(2\pi n/\omega_0, t)) = 0$, and can be written as [(2.17) of Section 2]

$$\mathrm{Re}\, f_0(s, t) = \gamma(t, s) \sin \omega_0 t + (1 - \cos \omega_0 t)\, \delta(s, t) \tag{C.2a}$$

where

$$\sup_s |\gamma(s, t)| = M < \infty \tag{C.2b}$$

and

$$\sup_s (\delta(s, t)) = N < \infty \tag{C.2c}$$

The real part of the term linear in s in the development of $-\frac{1}{2}[|\alpha(s)|^2 + |\alpha(s+t)|^2] + e^{it\omega_0}\bar{\alpha}(s)\,\alpha(t+s)$ furnishes

$$\begin{aligned}
&(1-\cos t\omega_0)\left(\frac{s\lambda_1^2}{4\omega_0}\cos 2s\omega_0 + \frac{\lambda_1^2 s}{4\omega_0}\cos 2\omega_0(s+t)\right.\\
&\quad+\frac{s\lambda_1\lambda_2}{\omega_0^2-\omega_2^2}\{\omega_0[\cos s\omega_2 \sin s\omega_0 + \cos\omega_2(s+t)\sin\omega_0 s - \omega_2]\\
&\quad\left.\times[\sin\omega_2(s+t)\sin\omega_0(s+t) + \sin\omega_2 s \sin\omega_0 s]\}\right)\\
&\quad+\sin t\omega_0\left(-\frac{\lambda_1^2 s}{4\omega_0}\sin 2s\omega_0 + \frac{\lambda_1^2 s}{4\omega_0}\sin 2\omega_0(s+t)\right.\\
&\quad+\frac{\lambda_1\lambda_2 s}{\omega_0^2-\omega_2^2}\{\omega_0[\cos\omega_2 \cos s\omega_0 + \cos\omega_2(s+t)\cos\omega_0(s+t)]\\
&\quad\left.-\omega_2[\sin\omega_2(s+t)\sin\omega_0(s+t) + \sin\omega_2 s \sin\omega_0 s]\}\right)
\end{aligned} \tag{C.3}$$

Hence (C.2) follows from (C.3), and

$$\operatorname{Re} f_0\left(s, \frac{2\pi n}{\omega_0}\right) = 0$$

is an immediate [(2.15) of the main text] consequence of (C.2a).

APPENDIX D

In the nonresonant case, $\alpha(s)$ in (2.10), (2.12) is given by

$$\begin{aligned}
\alpha(s) = &-[\omega_0^3\lambda_1 + \lambda_2\omega_0^3 - \lambda_2\omega_1^2\omega_0 - \lambda_1\omega_2^2\omega_0\\
&+e^{-is\omega_0}(\lambda_1\omega_0^3\cos s\omega_1)]/[(\omega_0^2-\omega_1^2)(\omega_0^2-\omega_2^2)]\\
&-e^{is\omega_0}[\lambda_1\omega_0\omega_2^2\cos s\omega_1 + \lambda_2\omega_0^3\cos s\omega_2 - \lambda_2\omega_0\omega_1^2\cos s\omega_2\\
&+i\lambda_1\omega_0^2\omega_1\sin s\omega_1]/[(\omega_0^2-\omega_1^2)(\omega_0^2-\omega_2^2)]\\
&+e^{-is\omega_0}[i\lambda_1\omega_1\omega_2^2\sin s\omega_1 + i\lambda_2\omega_0^2\omega_2\sin s\omega_2\\
&-i\lambda_2\omega_1^2\omega_1\sin s\omega_2]/[(\omega_0^2-\omega_1^2)(\omega_0^2-\omega_2^2)]
\end{aligned}$$

Therefore, in $I(t, s)$ given by (2.12), the arcs which arise as arguments of the trigonometric functions are of the form $(p\omega_0 \pm q\omega_1)$, $(r\omega_0 \pm m\omega_2)$, and $(v\omega_1 \pm u\omega_2)$, where p, q, r, m, u, and v are integers smaller than or equal to 3, and

$$I(t, s) = \sum_{j=1}^{N} v_j(s)\, u_j(t)$$

where $u_j(t)$ are almost-periodic [linear combinations of sines and cosines of $(\alpha_j t)$, where α_j is a linear function of the frequencies ω_1, ω_2, ω_3] and $v_j(s)$ is a sine or cosine of one of the previously mentioned arguments.

ACKNOWLEDGMENTS

One of us (W.F.W.) would like to thank L. Pasthur for pleasant discussions on related problems. We also thank the referees for important remarks which considerably improved the final version. H.H. was supported by FAPESP; W.F.W. was partially supported by CNPq.

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